# INFLUENCE OF TANGENTIAL ACCELERATION ON THE MOTION OF A SATELLITE

## (VLIIANIE KASATEL'NEGO USKORENIIA NA DVIZHENIE SPUTNIKA)

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Iu. G. EVTUSHENKO
(Moscow)

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The problem of satellite motion acted on by tangential acceleration generated by a low thrust engine was solved in [1 to 7]. Numerical solutions show [8], that the tangential control through the engine thrust approximates closely the solution of the problem of the optimal boost for a satellite.

We present here an asymptotic calculation of the influence of a small tangential acceleration on the motion of a satellite, using the method of averaging [9]. In section 1 we present a solution in the first approximation, valid for any elliptical orbit. In section 2, we show a solution in the second approximation for the case of circular orbits.

1. The system of equations for a plane motion of a satellite acted on by the tangential acceleration $f$, has the form

$$\frac{dZ}{dt} = \frac{2fp^{3/2}}{(1-e^2)^2\sqrt{\mu}}\sqrt{1+2a\cos u+2b\sin u+e^2}$$

$$\frac{du}{dt} = \sqrt{\mu}\,p^{-3/2}(1+a\cos u+b\sin u)^2 \qquad (1.1)$$

$$\frac{da}{dt} = \frac{2f\sqrt{p}\,(a+\cos u)}{\sqrt{\mu}\sqrt{1+2a\cos u+2b\sin u+e^2}}, \qquad \frac{db}{dt} = \frac{2f\sqrt{p}\,(b+\sin u)}{\sqrt{\mu}\sqrt{1+2a\cos u+2b\sin u+e^2}}$$

Here $\mu$ is the gravitational constant, $Z$ is the major semiaxis of the osculating ellipse, $u$ is the central angle between the position of the satellite and a certain fixed direction, $p = Z(1-e^2)$ is the focal parameter; the variables $a$ and $b$ are connected with the eccentricity of the orbit $e$ and with the angular orientation of the perigee $\delta$ through the formulas

$$e^2 = a^2 + b^2, \qquad \tan\sigma = ba^{-1}$$

We shall assume the small parameter to be $\varepsilon = fr_1^2\mu^{-1}$, which is the ratio of the rate of change of the thrust to the gravitational acceleration at some characteristic height $r_1$. Let us introduce the dimensionless major semiaxis $z$ and the dimensionless time of motion

$$\tau = t\sqrt{\mu}\,r_1^{-3/2}, \qquad z = Zr_1^{-1}$$

Then, the system (1.1) is reduced to the standard system with a fast rotating phase

$$\frac{dz}{d\tau} = \frac{2\varepsilon z^{3/2}}{\sqrt{1-e^2}} \sqrt{1 + 2a\cos u + 2b\sin u + e^2}, \qquad \frac{du}{d\tau} = \frac{(1 + a\cos u + b\sin u)^2}{[z(1-e^2)]^{3/2}}$$
$$\frac{da}{d\tau} = \frac{2\varepsilon(a+\cos u)\sqrt{z(1-e^2)}}{\sqrt{1+2a\cos u+2b\sin u+e^2}}, \qquad \frac{db}{d\tau} = \frac{2\varepsilon(b+\sin u)\sqrt{z(1-e^2)}}{\sqrt{1+2a\cos u+2b\sin u+e^2}} \tag{1.2}$$

In subsequent calculations we shall consider in the first place, that the ratio of the rate of change of the thrust to the gravitational acceleration is small over the whole interval of motion, and of the order of $0(\varepsilon)$, where $\varepsilon \ll 1$, and secondly, that, at any instant of time, the unperturbed motion (when $\varepsilon = 0$) is periodic. The fist condition limits the magnitude of thrust and the focal radius of the orbit ($r \sim r_1$), while the second condition is equivalent to the inequality $e < 1$ being rigorously fulfilled.

It has been shown in [6] that the system (1.2) has no singularities when $0 \leqslant e < 1$, and that it can be solved using the method of averaging. The variables $z$, $a$ and $b$ in the system (1.2) are slowly varying functions of time, while the function $u$ undergoes fast variation. Averaging the right-hand sides of the system (1.2) over a period of unperturbed (Kepplerian) motion $T = 2\pi z^{3/2}$, we obtain a system of equations in the first approximation

$$\frac{da}{d\tau} = \frac{4\varepsilon a\sqrt{z}(1-e^2)}{\pi e^2}[E(e) - K(e)], \qquad \frac{dz}{d\tau} = \frac{4}{\pi}\varepsilon z^{3/2} E(e)$$
$$\frac{db}{d\tau} = \frac{4\varepsilon b\sqrt{z}(1-e^2)}{\pi e^2}[E(e) - K(e)] \tag{1.3}$$

Here $K(e)$ and $E(e)$ are complete elliptic integrals of the first and second kind respectively, whose moduli are equal to the eccentricity of the orbit [10].

The solution of (1.3) approximates the exact solution of (1.2) with an error $\sim \varepsilon T$ in the interval $\tau \sim \varepsilon^{-1}$. First integrals the system (1.3) are

$$ab_0 = a_0 b, \qquad z = z_0 \frac{K(e_0) - E(e_0)}{K(e) - E(e)} \tag{1.4}$$

Here and in the following zero subscript denotes the initial values of the functions.

When $e \neq 0$ we can speak of the angular orientation of the line of apses and in this case we have from (1.4), that in the first approximation with respect to $\varepsilon$ the tangential acceleration does not change the angular orientation of the line of apses. It is easy to show that $K(e) \gg E(e)$ and the equality sign occurs only when $e = 0$. Consequently by (1.3) and (1.4) the major semi-axis of the osculating ellipse increases monotonely and the eccentricity decreases. Determination of the time of motion and the number of rotations of satellite can be reduced to quadratures

$$\tau = -\frac{\pi}{4\varepsilon\sqrt{z_0[K(e_0) - E(e_0)]}} \int_{e_0}^{e} \frac{e\,de}{(1-e^2)\sqrt{K-E}}$$
$$N = \int_0^\tau \frac{d\tau}{T} = -\frac{1}{8\varepsilon z_0^2[K(e_0) - E(e_0)]^2} \int_{e_0}^{e} \frac{e(K-E)}{1-e^2}de \tag{1.5}$$

In the case of orbits possessing small or large eccentricities the integrals (1.5) can be calculated from the known expansions of complete elliptic integrals into series with respect to the modulus or to an auxiliary modulus. For small eccentricities we find from (1.4) and (1.5)

$$z = \frac{z_0 e_0^2}{e^2} \frac{1 + {}^3/_8 e_0^2 + {}^{15}/_{64} e_0^4 + {}^{175}/_{1024} e_0^6 + {}^{2205}/_{16384} e_0^8}{1 + {}^3/_8 e^2 + {}^{15}/_{64} e^4 + {}^{175}/_{1024} e^6 + {}^{2205}/_{16384} e^8}$$

$$\tau = \frac{\sqrt{\pi}}{2\varepsilon \sqrt{z_0}\,[K(e_0) - E(e_0)]} [e_0 - e + {}^{13}/_{48} (e_0^3 - e^3) + {}^{383}/_{2560} (e_0^5 - e^5)] \qquad (1.6)$$

$$N = \frac{\pi\,[e_0^4 - e^4 + {}^{11}/_{12} (e_0^6 - e^6) + {}^{103}/_{128} (e_0^8 - e^8)]}{128\varepsilon z_0^2\,[K(e_0) - E(e_0)]^2}$$

The first terms of the expansion (1.6) were found previously by a number of authors (see for example [1]). We can investigate similarly the motion of a satellite with a slowly varying thrust. In this case it is sufficient to change in (1.5) the time-scale and the solutions (1.4) will remain valid.

2. When $e = 0$ then the averaged system (1.3) has the 'null' solution, a so called spiral trajectory, along which the eccentricity remains constant and the major semi-axis increases monotonely

$$e = 0, \qquad z = z_0 (1 - 4\varepsilon\tau \sqrt{z_0})^{-2} \qquad (2.1)$$

From (2.1) we can conclude, that if the initial eccentricity equals, or is nearly equal to zero then the exact solution for the eccentricity found by integrating the system (1.2), is a function of $T_1$ of the order $\sim O(\varepsilon)$ over the interval $\tau \sim \varepsilon^{-1}$. For a more detailed description of the evolution of the orbit, higher approximations are needed. The construction of a system in the second approximation is, for the general case, quite complicated, consequently we shall limit ourselves only to circular orbits. Let us introduce the new variable $L$

$$L = 2 \tan^{-1} \sqrt{\frac{1-e}{1+e}} \frac{a \sin u - b \cos u}{e + a \cos u + b \sin u} + \tan^{-1} \frac{b}{a} - \frac{\sqrt{1-e^2}\,(a \sin u - b \cos u)}{1 + a \cos u + b \sin u} \qquad (2.2)$$

Expanding (2.2) into series in $a$ and $b$, we obtain

$$u = L + 2a \sin L - 2b \cos L + O(e^2)$$

In this way, in the case of circular orbits when $a = b = 0$, the angle $L$ coincides with the angle of latitude $u$.

Differentiating the system (2.2) and using (1.2), we find

$$\frac{dL}{d\tau} = z^{-3/2} + \frac{a\dot{b} - b\dot{a}}{1 + \sqrt{1-e^2}} - \frac{2\varepsilon (1 - e^2) \sqrt{z}\,(a \sin u - b \cos u)}{(1 + a \cos u + b \sin u) \sqrt{1 + 2a \cos u + 2b \sin u + e^2}} \qquad (2.3)$$

Assuming that the initial eccentricity is small: $e_0 \sim O(\varepsilon)$, we expand the right-hand sides of (1.2) and (1.3) into a series with respect to $e$; neglecting the infinitesimals of the third order we obtain

$$\frac{dz}{d\tau} = 2\varepsilon z^{3/2} [1 + a \cos L + b \sin L], \qquad \frac{da}{d\tau} = \varepsilon \sqrt{z}\,[2 \cos L - a + a \cos 2L + b \sin 2L]$$
$$\frac{db}{d\tau} = \varepsilon \sqrt{z}\,[2 \sin L - b + a \sin 2L - b \cos 2L], \qquad \frac{dL}{d\tau} = z^{-3/2} - \varepsilon \sqrt{z}\,(a \sin L - b \cos L) \qquad (2.4)$$

Let us now perform the standard substitution of variables in the system (2.4)

$$z = D + \varepsilon z_1 + \varepsilon^2 z_2 + \ldots, \quad b = B + \varepsilon b_1 + \varepsilon^2 b_2 + \ldots$$
$$a = A + \varepsilon a_1 + \varepsilon^2 a_2 + \ldots, \quad L = \varphi + \varepsilon L_1 + \varepsilon^2 L_2 + \ldots \tag{2.5}$$

Here $z_1, a_1, b_1, L_1 \ldots$ are some functions of $D$, $A$, $B$ and $\varphi$. An algorithm for their determination is given in [9]. The functions $D$, $A$, $B$ and $\varphi$ in the first approximation satisfy the system

$$\frac{dD}{d\tau} = 2\varepsilon D^{3/2}, \qquad \frac{d\varphi}{d\tau} = D^{-3/2}, \qquad \frac{dA}{d\tau} = \frac{dB}{d\tau} = 0$$

which on integration gives

$$A = A_0, \quad B = B_0, \quad D = \frac{z_0}{(1 - 4\varepsilon\tau\sqrt{z_0})^2}, \quad \varphi = L_0 + \frac{D^2 - z_0^2}{4\varepsilon z_0^2 D^2}$$

For the functions $z_1$, $a_1$, $b_1$ and $L_1$ we shall assume that the zero harmonics with respect to $\varphi$, is absent, and using the general formulas for the determination of the functions $z_1$, $a_1$ $b_1$ and $L_1$ (see [9]), we shall obtain the formulas for the small-period perturbations of the elements of the orbit

$$z_1 = L_1 = 0, \qquad a_1 = 2D^2 \sin\varphi, \qquad b_1 = -2D^2 \cos\varphi \tag{2.6}$$

To find the solution in the second approximation we shall solve the averaged system (2.7)

$$\frac{dD}{d\tau} = 2\varepsilon D^{3/2}, \quad \frac{dB}{d\tau} = -\varepsilon B\sqrt{D}, \quad \frac{dA}{d\tau} = -\varepsilon A\sqrt{D}, \quad \frac{d\varphi}{d\tau} = D^{-3/2} - 2\varepsilon^2 D^{5/2}$$

with the initial conditions

$$D = z_0, \quad A = a_0 - \varepsilon a_{10}, \quad B = b_0 - \varepsilon b_{10}, \quad \varphi = L_0 \tag{2.8}$$

The solution of the system (2.7) has the form

$$D = \frac{z_0}{(1 - 4\varepsilon\tau\sqrt{z_0})^2}, \qquad A = F\left(\frac{z_0}{D}\right)^{1/2}, \qquad B = G\left(\frac{z_0}{D}\right)^{1/2}$$
$$\varphi = \varphi_0 + \frac{D^2 - z_0^2}{4\varepsilon z_0^2 D^2} + \frac{\varepsilon(z_0^2 - D^2)}{2}, \quad F = a_0 - 2\varepsilon z_0^2 \sin u_0, \quad G = b_0 + 2\varepsilon z_0^2 \cos u_0$$

Going back to the original variables we obtain

$$z = z_0(1 - 4\varepsilon\tau\sqrt{z_0})^{-2} \tag{2.9}$$

$$a = F\left(\frac{z_0}{z}\right)^{1/2} + 2\varepsilon z^2 \sin\left[u_0 + \frac{z^2 - z_0^2}{4\varepsilon z_0^2 z^2}\right] \tag{2.10}$$

$$b = G\left(\frac{z_0}{z}\right)^{1/2} - 2\varepsilon z^2 \cos\left[u_0 + \frac{z^2 - z_0^2}{4\varepsilon z_0^2 z^2}\right] \tag{2.11}$$

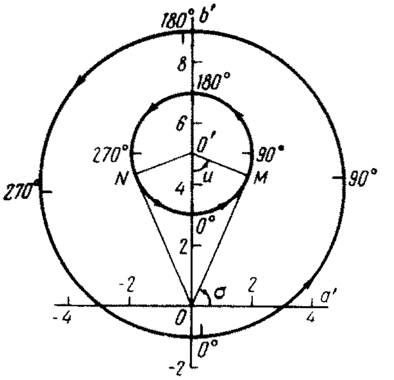
$$u = u_0 + \frac{z^2 - z_0^2}{4\varepsilon z^2 z_0^2} + 2a\sin\left(u_0 + \frac{z^2 - z_0^2}{4\varepsilon z^2 z_0^2}\right) + \frac{\varepsilon(z_0^2 - z^2)}{2} -$$
$$- 2b\cos\left(u_0 + \frac{z^2 - z_0^2}{4\varepsilon z^2 z_0^2}\right) + 2b_0 \cos u_0 - 2a_0 \sin u_0 \tag{2.12}$$

The solution (2.9) to (2.12) approximates the exact solution of the system (1.2) with the accuracy $\sim \varepsilon^2$ in the interval of time $\tau \sim \varepsilon^{-1}$, if $e_0 \sim O$ ($\varepsilon$).

From (2.9) it follows, that under the action of tangential acceleration the major semi-axis of the orbit is increases monotonely. The expressions for $a$ and $b$ have two terms each: the first term in each expression is a monotonely decreasing in absolute value function of time, while the second term oscillates with the frequency of the orbital motion of the satellite, the amplitude of these oscillations monotonely increasing.

To investigate the evolution of the eccentricity of the orbit, we shall introduce the quantity $E$ which is the mean value of the square of eccentricity over the period of one revolution of the satellite about a planet. By (2.10) and (2.11) we have

$$E = z_0 z^{-1} (F^2 + G^2) + 4\varepsilon^2 z^4 \qquad (2.13)$$




In the phase plane $a$, $b$ the solutions (2.10) and (2.11) give a parametric representation of the hodograph of the normalized Laplace vector (i.e. of the Laplace vector divided by $\mu$). Its modulus equals the eccentricity of the orbit, and the angle with the $a$-axis equals the angular distance of the perigee $\sigma$. The hodograph curve is a spiral with a slowly varying radius of curvature. When the value of $z$ is fixed, the equations (2.10) and (2.11) trace a circle of radius $2\varepsilon z^2$ , the coordinates of its center being $Fz_0^{1/2}z^{-1/2}$, $Gz_0^{1/2}z^{-1/2}$. As the time increases, the center of the circle moves toward the origin and its radius increases.

Two turns of the hodograph curve are shown in the Figure, where $a' = 10^4 a$, and $b' = 10^4 b$. These curves were obtained from the solution of the system (1.2) with the initial conditions $a_0 = 0$, $b_0 = 0.3 \times 10^{-3}$ , $u_0 = 0$, $z_0 = 1$, and $\varepsilon = 0.9 \times 10^{-3}$ when $r_1 = 3$. The smaller circle corresponds to the initial motion, the larger one corresponds to the interval of time $2038 < \tau < 2050$, and values of the angle $u$ over the interval $(0 - 2\pi)$ are marked on them. The arrows indicate the direction of motion of the Laplace vector.

At sufficiently small values of $z$, when the condition $b = Gz^{-1/2}z_0^{1/2} - 2\varepsilon z^2 \cos u > 0$, is satisfied, the hodograph lies in the upper half-plane, the line of apses performs oscillatory motions and the value of eccentricity oscillates about a certain mean value. As $z$ increases, the amplitude of vibrations of the line of apses increases although remaining smaller than $\pi_1$ and the mean value of the square of eccentricity monotonely increases. At the instant when the hodograph becomes tangent to the $a'$-axis, the eccentricity of the orbit becomes zero and the direction of the Laplace vector assumes the value $\pi$ in a discontinuous manner. The line of apses rotates with the same frequency as the satellite. At large values of $z$ the eccentricity monotonely increases $(e \sim 2\varepsilon z^2)$, the phase angle of the rotation of the line of apses differs by $\frac{1}{2}\pi$ from the phase of the central angle $u$; and in this case $\tan\sigma \approx -\cot u$. The motion of the satellite occurs in such a way, that it is always on a line perpendicular to the direction of the Laplace vector of the osculating ellipse.

From (2.9) to (2.11), we find the expression for the focal radius of the satellite

$$r = z - \sqrt{z_0 z}\,(F \cos u + G \sin u) \qquad (2.14)$$

Let us investigate the positions of a satellite at which the focal radius becomes maximum or minimum. They correspond to the apogee and perigee of the osculating ellipse. From the condition $\partial r/\partial u = 0$ we find the equation determining the value of $u$, which correspond to the extremal positions

$$2\varepsilon z^3 + \sqrt{zz_0}\,(F \sin u - G \cos u) = 0 \qquad (2.15)$$

Solutions of (2.15)

$$u = \pm \cos^{-1} \frac{2\varepsilon z^{5/2}}{\sqrt{z_0 (F^2 + G^2)}} - \cos^{-1} \left( \frac{G}{\sqrt{F^2 + G^2}} \right) \operatorname{sign} F \qquad (2.16)$$

exist, if

$$2\varepsilon z^{5/2} < \sqrt{z_0 (F^2 + G^2)} \qquad (2.17)$$

The plus sign in (2.16) corresponds to the perigee, while the minus sign to the apogee of the orbit. The condition (2.17) is equivalent to the geometric condition that the origin should be inside the hodograph curve. With the hodograph of the Laplace vector given, we can easily find the extrema geometrically. To do that we draw from the origin two lines tangent to the hodograph curve (on the figure they are the lines *OM* and *ON*). Since the angles $u$ and $\sigma$ are equal, it follows that the angular distance of the perigee equals the angle $OO'M$, and that of the apogee, $OO'N$.

As $z$ increases the amplitude of the vibrations of the radius decreases. The angle $NO'M$ between the apogee and perigee decreases and $e = 0$ it becomes zero. If the origin is inside the hodograph then the radius increases monotonely.

The condition (2.17) can be written as

$$4Z^2 f \sin (u - \sigma) < \mu e \qquad (2.18)$$

From (2.18) it follows, that if the ratio of the rate of change of thrust of the engine to the local gravitational acceleration is less than $e/4$, then the line of apses performs vibratory motions. At larger values of the perturbing acceleration, the line of apses rotates and the radius increases monotonely. The quantity $E$ is, in general, not a monotonic function of time. Differentiating (2.13) with respect to $u$, we find that $E$ decreases if

$$4\varepsilon z^2 < z_0^{1/2} z^{-1/2} \sqrt{F^2 + G^2} \qquad (2.19)$$

(geometrically it means, that the diameter of the circle traced by the hodograph of the Laplace vector is smaller than the distance between the center of the circle and the origin). The mean value of the square of eccentricity increases when the condition (2.19) is violated. In particular, if the initial orbit is a circle ($a_0 = b_0 = 0$) then the elements $E$, $\sigma$ and $r$ increase monotonely on the whole of the trajectory.

The approximate formulas obtained for the variations of the elements of the orbit, approximate the exact solution very closely. We present here some results of computations. The system (1.2) with initial conditions from the above example was solved numerically. It was found that when $\tau = 4255.086$, $z = 3.02994$, $e = 0.0021122$ and $u = 2227.687$, whereas the formulas (2.9) to (2.12) resulted in $z = 3.02993$, $e = 0.0021126$ and $u = 2227.687$.

The solution (2.14) for the focal radius was also obtained in [2] by another method.

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